Math 254B Lecture 24 Notes

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May 24, 2019

1 Moran's Theorem and Attractor Systems

1.1 Moran's theorem

Let $(\Phi_i)_{i=1}^k$ be an iterated function system with contraction coefficients r_i, $r^* = \max_i r_i$, $t_* = \min_i r_i \in (0,1)$. Last time, we showed that there exists a unique nonempty, compact attractor $K = \bigcup_{i=1}^k \Phi_i[K]$. We also had a coding map $\pi : [k]^{\mathbb{N}} \to K$ sending $\pi(\omega) = \lim_n \Phi_{\omega|_1^n}(x)$ for any x.

Theorem 1.1 (Moran)**.** *Let $k > 2$. Define $s > 0$ by $\sum_{i=1}^k r_i^s = 1$. Then* $\dim(K) \leq s$. *If the IFS satisfies the OSC,* $\dim(K) = s$.

Remark 1.1. The number s is sometimes called the **similarity dimension** $\operatorname{sdim}(\Phi)$.

Here is a heuristic: Assume $0 < m_\alpha(K) < 1$, and assume that the SSC holds. Then

$$m_\alpha(K) = \sum_{i=1}^k m_\alpha[\Phi_i[K]] = \sum_{i=1}^k r_i^\alpha m_\alpha(K).$$

Notation: Given $A \subseteq \mathbb{R}^d$, define $A_w = \Phi_w[A]$ for words $w \in [k]^n$. Then

$$\operatorname{diam}(A_w) = r_w \cdot \operatorname{diam}(A).$$

Proof. Upper bound: For all r, $K = \bigcup_{i=1}^k K_i = \bigcup_{i=1}^k \bigcup_{j=1}^k K_{i,j} = U_{w \in [k]^n} K_w$. These covers give:

$$\mathcal{H}^s_{\operatorname{diam}(K)(r^*)^n}(K) = \sum_{w \in [k]^n} (\operatorname{diam}(K_w))^s = \left[\sum_{w \in [k]^n} r_w^s\right] (\operatorname{diam}(K))^s = (\operatorname{diam}(K))^s.$$

So $m_s(K) < \infty$.

Lower bound: Let $\mu = (r_1^s, r_2^s, \dots, r_k^s)^{\times \mathbb{N}} \in P([k]^{\mathbb{N}})$, and let $\nu := \pi_* \mu$. We claim that ν is s-regular: $\nu(B_r(x)) \leq O(1) r^s$ for all $x \in K$.

Consider $[w] = \{w\omega : \omega[k]^{\mathbb{N}}\}$, the set of words that start with w. Then $\pi([w]) = K_w$. We have $\mu([w]) = r_{w_1}^s \cdots r_{w_n}^s = r_w^s$. On the other hand, $\operatorname{diam}(K_w) = O(1) r_w$. Choose $r > 0$ with $r \ll 1$. Define $\mathcal{W}_r := \{w \in \bigcup_n [k]^n : (\operatorname{diam}(K)) r_w < r, (\operatorname{diam}(K)) r_{w_1} \cdots r_{w_{n-1}} \geq r\}$. We know that $\bigcup_{w \in \mathcal{W}_r} [w] = [k]^{\mathbb{N}}$, and these $[w]$ are disjoint.

We claim that for any $x \in K$, $|\{w \in \mathcal{W}_n : K_w \cap B_r(x) \neq \varnothing\} = O_\Phi(1)$; that is, this depends on Φ but not on r or n. Then

$$\nu(B_r(x)) \leq \sum_{\substack{w \in \mathcal{W}_n \\ J_w \cap B_r(x) \neq \varnothing}} \mu([w]) \leq \sum_w O(1) \operatorname{diam}(K_w)^s \leq O_\Phi(1) r^s.$$

Let U be as in the open set condition, and pick a small ball $B = B(x_0, c_0) \subseteq U$. We know that $\Phi_i[U] \cap \Phi_j[U] = \varnothing$ if $i \neq j$, so $\Phi_i[B] \cap \Phi_j[B] = \varnothing$ if $i \neq j$. Consider words w, v. If w is an initial segment of v (written $w \leq v$), then $U_v \subseteq U_w$. If $w \not\leq v$ and $v \not\leq w$, then $U_w \cap U_v = \varnothing$, so $B_w \cap B_v = \varnothing$. In particular, $B_w \cap B_v = \varnothing$ if $w, v \in W_r$ are distinct. Moreover, $\operatorname{diam}(B_w) = \operatorname{diam}(B) r_w \sim r$ (up to a constant) if $w \in \mathcal{W}_r$. So if $B_r(x) \cap K_w \neq \varnothing$ and we know that $\operatorname{diam}(K_w) \sim r_w \sim r$, then there is some $C = O_\Phi(1)$ such that $B_{Cr}(x) \supseteq B_w$. This implies that $\{B_w : w \in \mathcal{W}_r, B_r(x) \cap K_w \neq \varnothing\}$ is a collection of disjoint balls contained in $B_{Cr}(x)$ with diameter $\sim r \sim Cr$. So the number of such balls is $\leq (\text{const})^d$. $\square$

Remark 1.2. If $r_i = r$ for all i, then $kr^s = 1$, so $s = \log(k)/\log(r^{-1})$. We will stay in this situation for most of the remaining results, but the results can be proven for more complicated IFSs (with more wrinkles in the proofs).

Remark 1.3. If $\Phi_i(x) = r_i U_i x + a_i$ for unitary U_i, we can construct self-affine attractors $\Phi_i = A_i x + b - i$, where $\|A_i\|_{\text{op}} < 1$. In this situation, the results in Moran's theorem give upper and lower bounds which generally don't match.

1.2 Attractor systems

How do the dynamics play into this picture? Let $\sigma : [k]^{\mathbb{N}} \to [k]^{\mathbb{N}}$ be the shift. $\sigma|_{[i]} = \psi_i^{-1}$, so we have local inverses. Under the SSC, we can define $S : K \to K$ siven by $S|_{K_i} = (\Phi_i|_K)^{-1}$

Definition 1.1. We call (K, S) the **attractor system**.

The coding map $\pi : ([k]^{\mathbb{N}}, \sigma) \to (K, S)$ is a conjugacy.

Theorem 1.2. *Let $r_i = r$ for all i. Under the SSC, for any ergodic $\mu \in P^S(K)$,*

$$\dim(\mu) = \frac{h(\mu, S)}{\log(r^{-1})}.$$

Proof. The sets $K_w = \pi([w])$, and $(\operatorname{diam}(K_w)) = \text{const} \cdot r^n$. Then

$$\nu(K_w) = \mu([w|_1^n]) = e^{-hn + o(n)} = (\operatorname{diam}(K_w))^{h/\log(r^{-1}) + o(1)}.$$

Run this idea with the construction in the proof of Moran's theorem. $\square$